

REAL-VARIABLE CHARACTERIZATIONS OF HARDY SPACES ASSOCIATED WITH BESSEL OPERATORS

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Let $\lambda > 0$, $p \in ((2\lambda + 1)/(2\lambda + 2), 1]$, and $\Delta_\lambda \equiv -\frac{d^2}{dx^2} - \frac{2\lambda}{x} \frac{d}{dx}$ be the Bessel operator. In this paper, the authors establish the characterizations of atomic Hardy spaces $H^p((0, \infty), dm_\lambda)$ associated with Δ_λ in terms of the radial maximal function, the nontangential maximal function, the grand maximal function, the Littlewood-Paley g -function and the Lusin-area function, where $dm_\lambda(x) \equiv x^{2\lambda} dx$. As an application, the authors further obtain the Riesz transform characterization of these Hardy spaces.

Keywords: Hardy space; Bessel operator; maximal function; Riesz transform; Littlewood-Paley g -function; Lusin-area function.

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1. Introduction

It is well known that the real-variable theory of Hardy spaces on the n -dimensional Euclidean space \mathbb{R}^n plays an important role in harmonic analysis and has been systematically developed; see [8, 9, 21, 22]. The classical Hardy spaces on \mathbb{R}^n are essentially related to the Laplacian $\Delta \equiv -\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$.

Let $\lambda \in (0, \infty)$ and Δ_λ be the Bessel operator, which is defined by setting, for all C^2 -functions f on $(0, \infty)$ and $x \in (0, \infty)$,

$$\Delta_\lambda f(x) \equiv -\frac{d^2}{dx^2} f(x) - \frac{2\lambda}{x} \frac{d}{dx} f(x).$$

In 1965, Muckenhoupt and Stein [19] developed a theory parallel to the classical case associated to Δ in the setting of Δ_λ , in which the results on $L^p((0, \infty), dm_\lambda)$ -boundedness of conjugate functions and fractional integrals associated with Δ_λ were obtained, where

$p \in [1, \infty)$ and $dm_\lambda(x) \equiv x^{2\lambda} dx$. Since then, many problems based on the Bessel context were studied; see, for example, [1, 2, 4, 5, 6, 16, 23, 24]. In particular, Betancor et al. in [3] established the characterizations of the atomic Hardy space $H^1((0, \infty), dm_\lambda)$ associated to Δ_λ in terms of the Riesz transform and the radial maximal function associated with the Hankel convolution of a class $Z^{[\lambda]}$ of functions, which includes the Poisson semigroup and the heat semigroup as special cases.

Let $p \in ((2\lambda+1)/(2\lambda+2), 1]$. The main target of this paper is, via using the results from [13, 14], to establish the characterizations of the atomic Hardy spaces $H^p((0, \infty), dm_\lambda)$ in terms of the radial maximal function, the nontangential maximal function, the grand maximal function, the Littlewood-Paley g -function and the Lusin-area function. As an application, we further obtain the Riesz transform characterization of these Hardy spaces.

To state our main results, we first recall some necessary notions and notation. Throughout this paper, we assume that $\lambda \in (0, \infty)$. Let Γ and J_ν respectively denote the *Gamma function* and the *Bessel function of the first kind of order ν* with $\nu \in (-1/2, \infty)$. For any f and $g \in L^1((0, \infty), dm_\lambda)$, their *Hankel convolution* is defined by setting, for all $x \in (0, \infty)$,

$$f \sharp_\lambda g(x) \equiv \int_0^\infty f(y) \tau_x^{[\lambda]} g(y) dm_\lambda(y), \quad (1.1)$$

where for $x \in (0, \infty)$, $\tau_x^{[\lambda]} g(y)$ denotes the *Hankel translation* of $g(y)$, that is,

$$\tau_x^{[\lambda]} g(y) \equiv \frac{\Gamma(\lambda + 1/2)}{\Gamma(\lambda) \sqrt{\pi}} \int_0^\pi g\left(\sqrt{x^2 + y^2 - 2xy \cos \theta}\right) (\sin \theta)^{2\lambda-1} d\theta. \quad (1.2)$$

In what follows, for any $x, r \in (0, \infty)$, let the *symbol* $I(x, r) \equiv (x - r, x + r) \cap (0, \infty)$. It is easy to see that

$$m_\lambda(I(x, r)) \sim \begin{cases} x^{2\lambda} r, & x > r; \\ r^{2\lambda+1}, & x \leq r. \end{cases} \quad (1.3)$$

This yields that

$$m_\lambda(I(x, r)) \sim x^{2\lambda} r + r^{2\lambda+1}, \quad (1.4)$$

which further implies that

$$2m_\lambda(I(x, r)) \lesssim m_\lambda(I(x, 2r)) \lesssim 2^{2\lambda+1} m_\lambda(I(x, r)). \quad (1.5)$$

Thus, $((0, \infty), \rho, dm_\lambda)$ is an RD-space introduced in [14], where $\rho(x, y) \equiv |x - y|$ for all $x, y \in (0, \infty)$. We now recall the notion of approximations of the identity in the context of RD-spaces, which was introduced in [14] (see also [13]).

Definition 1.1. Let $\epsilon_1 \in (0, 1]$ and $\epsilon_2, \epsilon_3 \in (0, \infty)$. A sequence $\{S_k\}_{k \in \mathbb{Z}}$ of bounded linear integral operators on $L^2((0, \infty), dm_\lambda)$ is called an *approximation of the identity of order $(\epsilon_1, \epsilon_2, \epsilon_3)$* (in short, $(\epsilon_1, \epsilon_2, \epsilon_3)$ - AOTI), if there exists a positive constant \tilde{C} such that for all $k \in \mathbb{Z}$ and $x, y \in (0, \infty)$, $S_k(x, y)$, the integral kernel of S_k is a measurable function from $(0, \infty) \times (0, \infty)$ into \mathbb{C} satisfying that

(i) for all $k \in \mathbb{Z}$ and $x, y \in (0, \infty)$,

$$|S_k(x, y)| \leq \tilde{C} \frac{1}{m_\lambda(I(x, 2^{-k})) + m_\lambda(I(y, 2^{-k})) + m_\lambda(I(x, |x - y|))} \frac{2^{-k\epsilon_2}}{(2^{-k} + |x - y|)^{\epsilon_2}};$$

(ii) for all $k \in \mathbb{Z}$ and $x, \tilde{x}, y \in (0, \infty)$ with $|x - \tilde{x}| \leq (2^{-k} + |x - y|)/2$,

$$|S_k(x, y) - S_k(\tilde{x}, y)| \leq \tilde{C} \frac{1}{m_\lambda(I(x, 2^{-k})) + m_\lambda(I(y, 2^{-k})) + m_\lambda(I(x, |x - y|))} \frac{|x - \tilde{x}|^{\epsilon_1} 2^{-k\epsilon_2}}{(2^{-k} + |x - y|)^{\epsilon_1 + \epsilon_2}};$$

(iii) property (ii) also holds with x and y interchanged;

(iv) for all $k \in \mathbb{Z}$ and $x, \tilde{x}, y, \tilde{y} \in (0, \infty)$ with $|x - \tilde{x}| \leq (2^{-k} + |x - y|)/3$ and $|y - \tilde{y}| \leq (2^{-k} + |x - y|)/3$,

$$|[S_k(x, y) - S_k(x, \tilde{y})] - [S_k(\tilde{x}, y) - S_k(\tilde{x}, \tilde{y})]| \leq \tilde{C} \frac{1}{m_\lambda(I(x, 2^{-k})) + m_\lambda(I(y, 2^{-k})) + m_\lambda(I(x, |x - y|))} \frac{|x - \tilde{x}|^{\epsilon_1} |y - \tilde{y}|^{\epsilon_1} 2^{-k\epsilon_3}}{(2^{-k} + |x - y|)^{2\epsilon_1 + \epsilon_3}};$$

(v) for all $k \in \mathbb{Z}$ and $x \in (0, \infty)$, $\int_0^\infty S_k(x, z) dm_\lambda(z) = 1 = \int_0^\infty S_k(z, x) dm_\lambda(z)$.

Remark 1.1. Similarly to [27, Remark 2.1(ii)], if a sequence $\{\tilde{S}_t\}_{t>0}$ of bounded linear integral operators on $L^2((0, \infty), dm_\lambda)$ satisfies (i) through (v) of Definition 1.1 with 2^{-k} replaced by t , then we call $\{\tilde{S}_t\}_{t>0}$ a *continuous approximation of the identity* of $(\epsilon_1, \epsilon_2, \epsilon_3)$ (in short, continuous $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI). For example, if $\{S_k\}_{k \in \mathbb{Z}}$ is an $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI and if we set $\tilde{S}_t(x, y) \equiv S_k(x, y)$ for $t \in (2^{-k-1}, 2^{-k}]$ with $k \in \mathbb{Z}$, then $\{\tilde{S}_t\}_{t>0}$ is a continuous $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI.

The following space of test functions was introduced in [14]; see also [13].

Definition 1.2. Let $x_1 \in (0, \infty)$, $r \in (0, \infty)$, $\beta \in (0, 1]$ and $\gamma \in (0, \infty)$. A function ϕ on $(0, \infty)$ is said to belong to the *space of test functions*, $\mathcal{G}(x_1, r, \beta, \gamma)$, if there exists a positive constant C such that

$$(G)_i \quad |\phi(x)| \leq C \frac{1}{m_\lambda(I(x, r+|x-x_1|))} \left(\frac{r}{r+|x-x_1|} \right)^\gamma \text{ for all } x \in (0, \infty);$$

$$(G)_{ii} \quad |\phi(x) - \phi(y)| \leq C \left(\frac{|x-y|}{r+|x_1-x|} \right)^\beta \frac{1}{m_\lambda(I(x, r+|x-x_1|))} \left(\frac{r}{r+|x_1-x|} \right)^\gamma \text{ for all } x, y \in (0, \infty)$$

satisfying that $|x - y| \leq (r + |x_1 - x|)/2$.

Moreover, for any $f \in \mathcal{G}(x_1, r, \beta, \gamma)$, its *norm* is defined by

$$\|f\|_{\mathcal{G}(x_1, r, \beta, \gamma)} \equiv \inf\{C : (G)_i \text{ and } (G)_{ii} \text{ hold}\}.$$

Remark 1.2. (i) Let $\{S_t\}_{t>0}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI for some positive constants ϵ_1, ϵ_2 and ϵ_3 , and $S_t(x, y)$ be the kernel of S_t . Obviously, $S_t(x, \cdot)$ for any fixed t and $x \in (0, \infty)$ is a test function of type $(x, t, \epsilon_1, \epsilon_2)$, and $S_t(\cdot, y)$ for any fixed t and $y \in (0, \infty)$ is a test function of type $(y, t, \epsilon_1, \epsilon_2)$; see also [14, p. 19].

(ii) For any $x \in (0, \infty)$, $1 + |x - 1| \sim 1 + x$. By this fact together with (1.4), if we take $x_1 \equiv 1$ and $r \equiv 1$ in Definition 1.2, we have that if $\phi \in \mathcal{G}(1, 1, \beta, \gamma)$, then

$$(G)_{i'} \quad |\phi(x)| \leq C \frac{1}{(1+x)^{2\lambda+1+\gamma}} \text{ for all } x \in (0, \infty);$$

$$(G)_{ii'} \quad |\phi(x) - \phi(y)| \leq C \frac{|x-y|^\beta}{(1+x)^{2\lambda+1+\gamma+\beta}} \text{ for all } x, y \in (0, \infty) \text{ satisfying that } |x - y| \leq (1 + |x - 1|)/2.$$

The space $\mathring{\mathcal{G}}(x, r, \beta, \gamma)$ is defined to be the set of all functions $f \in \mathcal{G}(x, r, \beta, \gamma)$ such that $\int_0^\infty f(x) dm_\lambda(x) = 0$. Moreover, we endow the space $\mathring{\mathcal{G}}(x, r, \beta, \gamma)$ with the same norm as the space $\mathcal{G}(x, r, \beta, \gamma)$.

The space $\mathcal{G}(x, r, \beta, \gamma)$ is a Banach space. Let $\epsilon \in (0, 1]$ and $\beta, \gamma \in (0, \epsilon]$. We further define the space $\mathcal{G}_0^\epsilon(x, r, \beta, \gamma)$ to be the completion of the set $\mathcal{G}(x, r, \epsilon, \epsilon)$ in $\mathcal{G}(x, r, \beta, \gamma)$. For $f \in \mathcal{G}_0^\epsilon(x, r, \beta, \gamma)$, define $\|f\|_{\mathcal{G}_0^\epsilon(x, r, \beta, \gamma)} \equiv \|f\|_{\mathcal{G}(x, r, \beta, \gamma)}$. Let $(\mathcal{G}_0^\epsilon(x, r, \beta, \gamma))'$ be the set of all continuous linear functionals on $\mathcal{G}_0^\epsilon(x, r, \beta, \gamma)$ endowed with the weak* topology, and denote by $\langle f, \varphi \rangle$ the natural pairing of elements $f \in (\mathcal{G}_0^\epsilon(x, r, \beta, \gamma))'$ and $\varphi \in \mathcal{G}_0^\epsilon(x, r, \beta, \gamma)$. Throughout this paper, we fix $x_1 \equiv 1$ and write $\mathcal{G}_0^\epsilon(\beta, \gamma) \equiv \mathcal{G}_0^\epsilon(1, 1, \beta, \gamma)$, and $(\mathcal{G}_0^\epsilon(\beta, \gamma))' \equiv (\mathcal{G}_0^\epsilon(1, 1, \beta, \gamma))'$.

Similarly, let the space $\mathring{\mathcal{G}}_0^\epsilon(x, r, \beta, \gamma)$ be the completion of the set $\mathring{\mathcal{G}}(x, r, \epsilon, \epsilon)$ in the space $\mathring{\mathcal{G}}(x, r, \beta, \gamma)$. For any $f \in \mathring{\mathcal{G}}_0^\epsilon(x, r, \beta, \gamma)$, define $\|f\|_{\mathring{\mathcal{G}}_0^\epsilon(x, r, \beta, \gamma)} \equiv \|f\|_{\mathring{\mathcal{G}}(x, r, \beta, \gamma)}$. Denote by $(\mathring{\mathcal{G}}_0^\epsilon(x, r, \beta, \gamma))'$ the space of all continuous linear functionals on $\mathring{\mathcal{G}}_0^\epsilon(x, r, \beta, \gamma)$, and endow $(\mathring{\mathcal{G}}_0^\epsilon(x, r, \beta, \gamma))'$ with the weak* topology. We always write $\mathring{\mathcal{G}}_0^\epsilon(\beta, \gamma) \equiv \mathring{\mathcal{G}}_0^\epsilon(1, 1, \beta, \gamma)$. See [14] or [13] for the details. Moreover, it was proved in [28] that for any $\epsilon, \tilde{\epsilon} \in (0, 1)$ and $\beta, \gamma \in (0, \min\{\epsilon, \tilde{\epsilon}\})$, the spaces $\mathcal{G}_0^\epsilon(\beta, \gamma) = \mathcal{G}_0^{\tilde{\epsilon}}(\beta, \gamma)$ and $\mathring{\mathcal{G}}_0^\epsilon(\beta, \gamma) = \mathring{\mathcal{G}}_0^{\tilde{\epsilon}}(\beta, \gamma)$.

We now recall the atomic Hardy spaces $H^p((0, \infty), dm_\lambda)$ in [14]; see also [7].

Definition 1.3. Let $p \in ((2\lambda + 1)/(2\lambda + 2), 1]$, $\epsilon \in (0, 1)$ and $\beta, \gamma \in (0, \epsilon)$. A function a is called an $H^p((0, \infty), dm_\lambda)$ -atom if there exists an open bounded interval $I \subset (0, \infty)$ such that $\text{supp}(a) \subset I$, $\|a\|_{L^2((0, \infty), dm_\lambda)} \leq [m_\lambda(I)]^{1/2-1/p}$ and $\int_0^\infty a(x) dm_\lambda(x) = 0$.

A distribution $f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))'$ is said to belong to the Hardy space $H^p((0, \infty), dm_\lambda)$ if $f = \sum_{j=1}^\infty \alpha_j a_j$ in $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$, where for every j , a_j is an $H^p((0, \infty), dm_\lambda)$ -atom and $\alpha_j \in \mathbb{C}$ satisfying that $\sum_{j=1}^\infty |\alpha_j|^p < \infty$. The norm $\|f\|_{H^p((0, \infty), dm_\lambda)}$ of f in $H^p((0, \infty), dm_\lambda)$ is defined by $\|f\|_{H^p((0, \infty), dm_\lambda)} \equiv \inf\{(\sum_{j=1}^\infty |\alpha_j|^p)^{1/p}\}$, where the infimum is taken over all the decompositions of f as above.

The following class $Z^{[\lambda]}$ of functions is a slight variant of the corresponding class introduced in [3].

Definition 1.4. Let $Z^{[\lambda]}$ be the set of all C^2 -functions ϕ on $[0, \infty)$ such that for all $x \geq 0$,

$$0 \leq \phi(x) \leq C(1 + x^2)^{-\lambda-1}, \quad (1.6)$$

$$|\phi'(x)| \leq Cx(1 + x^2)^{-\lambda-2}, \quad (1.7)$$

and

$$|\phi''(x)| \leq C(1 + x^2)^{-\lambda-2}. \quad (1.8)$$

We now recall the radial maximal function, the nontangential maximal function, the grand maximal function, the Littlewood-Paley g -function and the Lusin-area function in [14].

Definition 1.5. Let $\phi \in Z^{[\lambda]}$, $\epsilon \in (0, 1]$ and $\beta, \gamma \in (0, \epsilon)$. For any $f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))'$, the radial maximal function, the nontangential maximal function and the grand maximal function are defined by setting, for all $x \in (0, \infty)$,

$$\Phi^+(f)(x) \equiv \sup_{t>0} |\Phi_t(f)(x)| \equiv \sup_{t>0} |f \sharp_\lambda \phi_t(x)|,$$

$$\Phi^*(f)(x) \equiv \sup_{t>0, |x-y|<t} |\Phi_t(f)(y)| \equiv \sup_{t>0, |x-y|<t} |f \#_\lambda \phi_t(y)|$$

and

$$G^{(\epsilon, \beta, \gamma)}(f)(x) \equiv \sup\{\langle f, \varphi \rangle : \varphi \in \mathcal{G}_0^\epsilon(\beta, \gamma), \|\varphi\|_{\mathcal{G}(x, r, \beta, \gamma)} \leq 1 \text{ for some } r \in (0, \infty)\},$$

where for all $t, y \in (0, \infty)$, $\phi_t(y) \equiv t^{-2\lambda-1}\phi(y/t)$.

Remark 1.3. Observe that the functions

$$P^{[\lambda]}(x) \equiv \frac{2\lambda\Gamma(\lambda)}{\Gamma(\lambda+1/2)\sqrt{\pi}} \frac{1}{(1+x^2)^{\lambda+1}}$$

and $W^{[\lambda]}(x) \equiv 2^{(1-2\lambda)/2} \exp(-x^2/2) / \Gamma(\lambda+1/2)$ both belong to $Z^{[\lambda]}$. Recall that $P_t^{[\lambda]}(f) \equiv e^{-t\sqrt{\Delta_\lambda}} f = f \#_\lambda P_t^{[\lambda]}$ and $W_t^{[\lambda]}(f) \equiv e^{-t\Delta_\lambda} f = f \#_\lambda W_{\sqrt{2t}}^{[\lambda]}$ (see [3, pp. 200-201]). Thus, the radial maximal functions and the nontangential maximal functions respectively associated with $\{e^{-t\sqrt{\Delta_\lambda}}\}_{t>0}$ and $\{e^{-t\Delta_\lambda}\}_{t>0}$ are special cases of Φ^+ and Φ^* .

Definition 1.6. Let $\epsilon_1 \in (0, 1]$, $\epsilon_2, \epsilon_3 \in (0, \infty)$, $a \in (0, \infty)$, $\epsilon \in (0, \min\{\epsilon_1, \epsilon_2\})$, $\beta, \gamma \in (0, \epsilon)$ and $\{S_k\}_{k \in \mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$ -AOTI. For all $f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))'$, the *Littlewood-Paley g -function* $g(f)$ and the *Lusin-area function* $S_a(f)$ are respectively defined by setting, for all $x \in (0, \infty)$,

$$g(f)(x) \equiv \left[\sum_{k \in \mathbb{Z}} |S_k(f)(x) - S_{k-1}(f)(x)|^2 \right]^{1/2}$$

and

$$S_a(f)(x) \equiv \left\{ \sum_{k \in \mathbb{Z}} \int_{|x-y|<a2^{-k}} |S_k(f)(y) - S_{k-1}(f)(y)|^2 \frac{dm_\lambda(y)}{m_\lambda(I(x, a2^{-k}))} \right\}^{1/2}.$$

The first result of this paper is as follows.

Theorem 1.1. Let $\lambda \in (0, \infty)$, $p \in ((2\lambda+1)/(2\lambda+2), 1]$, $\epsilon_1 \in (0, 1]$, $\epsilon_2, \epsilon_3 \in (0, \infty)$, $a \in (0, \infty)$, $\epsilon \in (0, \min\{\epsilon_1, \epsilon_2\})$, $\beta, \gamma \in ((2\lambda+1)(1/p-1), \epsilon)$ and $\phi \in Z^{[\lambda]}$. A distribution $f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))'$ belongs to $H^p((0, \infty), dm_\lambda)$ if and only if $\mathcal{M}(f) \in L^p((0, \infty), dm_\lambda)$; moreover,

$$\|f\|_{H^p((0, \infty), dm_\lambda)} \sim \|\mathcal{M}(f)\|_{L^p((0, \infty), dm_\lambda)},$$

where $\mathcal{M}(f)$ is one of $\Phi^+(f)$, $\Phi^*(f)$, $G^{(\epsilon, \beta, \gamma)}(f)$, $g(f)$ and $S_a(f)$.

Differently from [3], we establish Theorem 1.1 by first showing that $\{\Phi_t\}_{t>0}$ defined as in Definition 1.5 is actually a constant multiple of an approximation of the identity as in Definition 1.1; see Lemma 2.1 below. We then obtain all desired conclusions of Theorem 1.1 by directly applying results in [14, 10, 27]. The details are given in Section 2.

By applying Theorem 1.1, we next establish the characterization of $H^p((0, \infty), dm_\lambda)$ in terms of the Riesz transform R_{Δ_λ} . Let $r \in [1, \infty)$ and $f \in L^r((0, \infty), dm_\lambda)$. The Δ_λ -conjugate of f is defined by setting, for any $t, x \in (0, \infty)$,

$$Q_t^{[\lambda]}(f)(x) \equiv \int_0^\infty Q_t^{[\lambda]}(x, y) f(y) dm_\lambda(y),$$

where for any $t, x, y \in (0, \infty)$,

$$\begin{aligned} Q_t^{[\lambda]}(x, y) &\equiv -(xy)^{-\lambda+1/2} \int_0^\infty e^{-t\xi} \xi J_{\lambda+1/2}(x\xi) J_{\lambda-1/2}(\xi y) d\xi \\ &= -\frac{2\lambda}{\pi} \int_0^\pi \frac{(x - y \cos \theta)(\sin \theta)^{2\lambda-1}}{(x^2 + y^2 + t^2 - 2xy \cos \theta)^{\lambda+1}} d\theta; \end{aligned} \quad (1.9)$$

see [19, p.84]. Moreover, there exists the boundary value function $\lim_{t \rightarrow 0} Q_t^{[\lambda]}(f)(x)$ for almost every $x \in (0, \infty)$ (see [19, p.84]), which is defined to be the *Riesz transform* $R_{\Delta_\lambda}(f)$. Muckenhoupt and Stein [19, p.87] also proved that R_{Δ_λ} is bounded on $L^r((0, \infty), dm_\lambda)$ when $r \in (1, \infty)$. In [4, pp.710-711], Betancor et al. further showed that if $r \in [1, \infty)$ and $f \in L^r((0, \infty), dm_\lambda)$, then for almost every $x \in (0, \infty)$,

$$R_{\Delta_\lambda}(f)(x) = \lim_{\delta \rightarrow 0} \int_{0, |x-y|>\delta}^\infty Q_0^{[\lambda]}(x, y) f(y) dm_\lambda(y),$$

where for any $x, y \in (0, \infty)$,

$$Q_0^{[\lambda]}(x, y) \equiv -\frac{2\lambda}{\pi} \int_0^\pi \frac{(x - y \cos \theta)(\sin \theta)^{2\lambda-1}}{(x^2 + y^2 - 2xy \cos \theta)^{\lambda+1}} d\theta.$$

Moreover, Betancor, Fariña and Sanabria [5] showed that R_{Δ_λ} is a Calderón-Zygmund operator on the space $((0, \infty), \rho, dm_\lambda)$ of homogeneous type, where R_{Δ_λ} is bounded from $H^1((0, \infty), dm_\lambda)$ to $L^1((0, \infty), dm_\lambda)$.

Let $\epsilon \in (0, 1]$, $\beta, \gamma, \tilde{\epsilon} \in (0, \epsilon)$ and $f \in (\mathcal{G}_0^{\tilde{\epsilon}}(\beta, \gamma))'$. $R_{\Delta_\lambda}(f)$ is said to belong to $(\mathring{\mathcal{G}}(\beta, \gamma))'$, if there exists $F \in (\mathring{\mathcal{G}}_0^{\tilde{\epsilon}}(\beta, \gamma))'$ such that for all $\psi \in \mathring{\mathcal{G}}_0^{\tilde{\epsilon}}(\beta, \gamma)$,

$$\langle R_{\Delta_\lambda}(f), \psi \rangle \equiv \int_0^\infty f(x) \tilde{R}_{\Delta_\lambda}(\psi)(x) dm_\lambda(x) = \int_0^\infty F(x) \psi(x) dm_\lambda(x),$$

where $\tilde{R}_{\Delta_\lambda}$ is the adjoint operator of R_{Δ_λ} ; see also [3, Lemma 2.42]. By Lemma 3.1 below, we see that $\tilde{R}_{\Delta_\lambda}$ is bounded from $\mathring{\mathcal{G}}_0^{\tilde{\epsilon}}(\beta, \gamma)$ to $\mathring{\mathcal{G}}_0^{\tilde{\epsilon}}(\beta, \gamma)$.

A distribution $f \in (\mathcal{G}_0^{\tilde{\epsilon}}(\beta, \gamma))'$ is said to be *restricted at infinity*, if for any $\phi \in Z^{[\lambda]}$ and $r > 0$ large enough, $f \sharp_\lambda \phi \in L^r((0, \infty), dm_\lambda)$. By Theorem 1.1 and an argument as in [21, pp.100-101], we see that for any $f \in H^p((0, \infty), dm_\lambda)$ with $p \in ((2\lambda + 1)/(2\lambda + 2), 1]$ and $\phi \in Z^{[\lambda]}$, $f \sharp_\lambda \phi \in L^r((0, \infty), dm_\lambda)$ for all $r \in [p, \infty]$. Moreover, we have the following characterization of $H^p((0, \infty), dm_\lambda)$ in terms of the Riesz transform R_{Δ_λ} .

Theorem 1.2. *Let $\lambda \in (0, \infty)$, $p \in ((2\lambda + 1)/(2\lambda + 2), 1]$, $\epsilon \in (0, 1]$, $\beta, \gamma \in ((2\lambda + 1)(1/p - 1), \epsilon)$, $\tilde{\epsilon} \in (0, \epsilon)$, $\phi \in Z^{[\lambda]}$ and $f \in (\mathring{\mathcal{G}}_0^{\tilde{\epsilon}}(\beta, \gamma))'$ be restricted at infinity. Then $f \in H^p((0, \infty), dm_\lambda)$ if and only if there exists a positive constant C such that for all $\delta \in (0, \infty)$,*

$$\|f \sharp_\lambda \phi \delta\|_{L^p((0, \infty), dm_\lambda)} + \|R_{\Delta_\lambda}(f \sharp_\lambda \phi \delta)\|_{L^p((0, \infty), dm_\lambda)} \leq C. \quad (1.10)$$

To show the necessity of Theorem 1.2, by using the molecular characterization of the atomic Hardy space $H^p((0, \infty), dm_\lambda)$ (see Theorems 2.1 and 2.2 in [15]) and the boundedness criterion on sublinear operators on Hardy spaces over RD-spaces (see [26] or [10]), we first show that the Riesz transform R_{Δ_λ} is bounded from $H^p((0, \infty), dm_\lambda)$ to $L^p((0, \infty), dm_\lambda)$ (see Lemma 3.2 below) and $f\sharp_\lambda\phi_\delta$ is bounded on $H^p((0, \infty), dm_\lambda)$ uniformly on $\delta \in (0, \infty)$, which further induces the necessity of Theorem 1.2.

To show the sufficiency of Theorem 1.2, we establish a key estimate for the radial maximal function of the first entry of the conjugate harmonic systems satisfying the generalized Cauchy-Riemann equations associated with Δ_λ in terms of maximal L^p norm of the conjugate harmonic systems (see Lemma 3.4 below), which when $p = 1$ was already obtained by Betancor et al. in [3, Lemma 2.38]. Then, as an application of Theorem 1.1, we obtain the sufficiency of Theorem 1.2. The details are given in Section 3.

We remark that (1.10) is formally slightly different from the case for $H^p(\mathbb{R}^n)$ (see [21, p.123]). Recall that a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ restricted at infinity belongs to $H^p(\mathbb{R}^n)$ with $p \in ((n-1)/n, 1]$ if and only if there exists a positive constant C such that for all $\delta \in (0, \infty)$,

$$\|f * \phi_\delta\|_{L^p(\mathbb{R}^n)} + \sum_{j=1}^n \|R_j(f) * \phi_\delta\|_{L^p(\mathbb{R}^n)} \leq C, \quad (1.11)$$

where $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\phi_\delta(x) \equiv \delta^{-n}\phi(x/\delta)$, and $\{R_j\}_{j=1}^n$ are the classical Riesz transforms; see [21, p.123]. Since $\{R_j\}_{j=1}^n$ are convolution operators, we have that for all $j \in \{1, \dots, n\}$,

$$R_j(f) * \phi_\delta = (K_j * f) * \phi_\delta = K_j * (f * \phi_\delta) = R_j(f * \phi_\delta),$$

where K_j is the kernel of R_j . Thus, for $H^p(\mathbb{R}^n)$ with $p \in ((n-1)/n, 1]$, (1.11) and (1.10) are the same, and actually these commutative relations were used in the proof of [21, p.123, Proposition 3]. However, it is unclear if this is also true for the Hardy space $H^p((0, \infty), dm_\lambda)$. Nevertheless, from Theorems 1.1 and 1.2 together with the boundedness of R_{Δ_λ} on $H^p((0, \infty), dm_\lambda)$ (see Lemma 3.2 below), we immediately deduce the following result. The details are omitted.

Corollary 1.1. *Let $\lambda \in (0, \infty)$, $p \in ((2\lambda+1)/(2\lambda+2), 1]$, $\epsilon \in (0, 1]$, $\beta, \gamma \in ((2\lambda+1)(1/p-1), \epsilon)$, $\tilde{\epsilon} \in (0, \epsilon)$ and $f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))'$ be restricted at infinity. Then $f \in H^p((0, \infty), dm_\lambda)$ if and only if there exists a positive constant \tilde{C} such that for all $\delta \in (0, \infty)$,*

$$\|f\sharp_\lambda\phi_\delta\|_{L^p((0, \infty), dm_\lambda)} + \|R_{\Delta_\lambda}(f\sharp_\lambda\phi_\delta)\|_{L^p((0, \infty), dm_\lambda)} + \|(R_{\Delta_\lambda}(f))\sharp_\lambda\phi_\delta\|_{L^p((0, \infty), dm_\lambda)} \leq \tilde{C}.$$

Finally, we make some conventions on notation. Throughout the paper, we denote by C and \tilde{C} positive constants which are independent of the main parameters, but they may vary from line to line. If $f \leq Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$. For any $k \in (0, \infty)$ and $I \equiv I(x, r)$ for some $x, r \in (0, \infty)$, $kI \equiv (x - kr, x + kr) \cap (0, \infty)$.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We start with the following key lemma.

Lemma 2.1. *Let $\phi \in Z^{[\lambda]}$ and $\{\Phi_t\}_{t>0}$ be as in Definition 1.5. Then*

$$\left\{ \frac{1}{\|\phi\|_{L^1((0,\infty), dm_\lambda)}} \Phi_t \right\}_{t>0}$$

is a continuous $(1, 1, 1)$ – AOTI.

Proof. By Definition 1.5 and (1.1), we see that for all $t, x, y \in (0, \infty)$, the kernel $\Phi_t(x, y) \equiv \tau_x^{[\lambda]} \phi_t(y)$. For all $x, y, z \in (0, \infty)$, let $\Delta(x, y, z)$ be the area of a triangle with sides x, y, z when such a triangle exists, and

$$D(x, y, z) \equiv \frac{2^{2\lambda-2} \Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda) \sqrt{\pi}} (xyz)^{-2\lambda+1} [\Delta(x, y, z)]^{2\lambda-2}$$

when such $\Delta(x, y, z)$ exists, and zero otherwise. Then by (1.2) and the change of variables, we obtain that

$$\begin{aligned} \Phi_t(x, y) &= \frac{\Gamma(\lambda + 1/2)}{\Gamma(\lambda) \sqrt{\pi}} \int_0^\pi \phi_t \left(\sqrt{x^2 + y^2 - 2xy \cos \theta} \right) (\sin \theta)^{2\lambda-1} d\theta \\ &= \frac{\Gamma(\lambda + 1/2)}{\Gamma(\lambda) \sqrt{\pi}} \int_{|x-y|}^{x+y} \phi_t(z) (xy)^{-1} z \left[1 - \left(\frac{x^2 + y^2 - z^2}{2xy} \right)^2 \right]^{\lambda-1} dz \\ &= \int_0^\infty \phi_t(z) D(x, y, z) dm_\lambda(z). \end{aligned} \quad (2.1)$$

From this together with (6) in [12, p. 335] and the change of variables, we further deduce that for all $x, t \in (0, \infty)$,

$$\int_0^\infty \Phi_t(x, y) dm_\lambda(y) = \int_0^\infty \int_0^\infty \phi_t(z) D(x, y, z) dm_\lambda(y) dm_\lambda(z) = \|\phi\|_{L^1((0,\infty), dm_\lambda)}. \quad (2.2)$$

By the homogeneity of $L^1((0, \infty), dm_\lambda)$, we may assume that $\|\phi\|_{L^1((0,\infty), dm_\lambda)} = 1$. Then by (1.2) and (2.2), $\Phi_t(x, y)$ is symmetric in x and y , and satisfies (v) of Definition 1.1. Moreover, it follows from (2.2) that $\{\Phi_t\}_{t>0}$ is uniformly bounded on both $L^1((0, \infty), dm_\lambda)$ and $L^\infty((0, \infty), dm_\lambda)$, which together with the Marcinkiewicz interpolation theorem yields that $\{\Phi_t\}_{t>0}$ is also uniformly bounded on $L^2((0, \infty), dm_\lambda)$. This can also be deduced from (2.2), the Hölder inequality, the symmetry of $\Phi_t(x, y)$ and the Fubini theorem as follows: for all $f \in L^2((0, \infty), dm_\lambda)$,

$$\begin{aligned} \int_0^\infty \left| \int_0^\infty \Phi_t(x, y) f(y) dm_\lambda(y) \right|^2 dm_\lambda(x) &\leq \int_0^\infty \int_0^\infty \Phi_t(x, y) |f(y)|^2 dm_\lambda(y) dm_\lambda(x) \\ &= \int_0^\infty |f(y)|^2 dm_\lambda(y). \end{aligned}$$

Thus, by the symmetry of $\Phi_t(x, y)$, to finish the proof of Lemma 2.1, we still need to show that $\{\Phi_t\}_{t>0}$ satisfies (i), (ii) and (iv) of Definition 1.1. We first prove that $\{\Phi_t\}_{t>0}$ satisfies Definition 1.1(i). By (1.4), we obtain that for all $x, y, t \in (0, \infty)$,

$$m_\lambda(I(y, t)) \lesssim m_\lambda(I(x, t)) + m_\lambda(I(x, |x - y|)). \quad (2.3)$$

Then Definition 1.1(i) is reduced to showing that for all $x, y, t \in (0, \infty)$,

$$\Phi_t(x, y) \lesssim \frac{1}{m_\lambda(I(x, t)) + m_\lambda(I(x, |x - y|))} \frac{t}{t + |x - y|}. \quad (2.4)$$

To this end, by (1.2) and (1.6), we see that

$$\begin{aligned} \Phi_t(x, y) &\sim t^{-2\lambda-1} \int_0^\pi \phi\left(\frac{\sqrt{x^2 + y^2 - 2xy \cos \theta}}{t}\right) (\sin \theta)^{2\lambda-1} d\theta \\ &\lesssim t^{-2\lambda-1} \int_0^\pi \left(1 + \frac{x^2 + y^2 - 2xy \cos \theta}{t^2}\right)^{-\lambda-1} (\sin \theta)^{2\lambda-1} d\theta \\ &\lesssim \int_0^\pi \frac{t(\sin \theta)^{2\lambda-1}}{(t^2 + x^2 + y^2 - 2xy \cos \theta)^{\lambda+1}} d\theta. \end{aligned} \quad (2.5)$$

We then consider the following two cases.

Case (i) $t \geq x$ or $|x - y| \geq x/2$. In this case, from (2.5) and the fact that

$$\int_0^\pi (\sin \theta)^{2\lambda-1} d\theta = \frac{\Gamma(\lambda)\sqrt{\pi}}{\Gamma(\lambda + \frac{1}{2})}, \quad (2.6)$$

we deduce that

$$\Phi_t(x, y) \lesssim \frac{t}{(t^2 + |x - y|^2)^{\lambda+1}},$$

which together with (1.4) yields (2.4).

Case (ii) $t < x$ and $|x - y| < x/2$. In this case, (2.4) follows from

$$\Phi_t(x, y) \lesssim \frac{t}{x^{2\lambda}(t + |x - y|)^2}. \quad (2.7)$$

Observe that in this case, $x \sim y$. Then by the fact that for all $\theta \in (0, \pi/2]$, $\sin \theta \sim \theta$ and $1 - \cos \theta \geq 2(\theta/\pi)^2$, we have

$$\begin{aligned} E_1 &\equiv \int_0^{\pi/2} \frac{t(\sin \theta)^{2\lambda-1}}{[t^2 + |x - y|^2 + 2xy(1 - \cos \theta)]^{\lambda+1}} d\theta \\ &\lesssim \int_0^{\pi/2} \frac{t\theta^{2\lambda-1}}{[t^2 + |x - y|^2 + 4xy\theta^2/\pi^2]^{\lambda+1}} d\theta \\ &\lesssim \frac{t}{(xy)^\lambda(t^2 + |x - y|^2)} \int_0^\infty \frac{\beta^{2\lambda-1}}{(1 + \beta^2)^{\lambda+1}} d\beta \lesssim \frac{t}{x^{2\lambda}(t^2 + |x - y|^2)}. \end{aligned}$$

On the other hand, by (2.6) and the fact that $\cos \theta < 0$ for all $\theta \in (\pi/2, \pi]$, we have that

$$\begin{aligned} E_2 &\equiv \int_{\pi/2}^\pi \frac{t(\sin \theta)^{2\lambda-1}}{(t^2 + x^2 + y^2 - 2xy \cos \theta)^{\lambda+1}} d\theta \\ &\lesssim \frac{t}{x^{2\lambda}(t^2 + |x - y|^2)} \int_{\pi/2}^\pi (\sin \theta)^{2\lambda-1} d\theta \lesssim \frac{t}{x^{2\lambda}(t^2 + |x - y|^2)}, \end{aligned}$$

which together with the estimate of E_1 yields (2.7).

We now show that $\{\Phi_t\}_{t>0}$ satisfies Definition 1.1(ii). By (2.3), it suffices to show that for all $x, \tilde{x}, y, t \in (0, \infty)$ with $|x - \tilde{x}| \leq (t + |x - y|)/2$,

$$F \equiv |\Phi_t(x, y) - \Phi_t(\tilde{x}, y)| \lesssim \frac{1}{m_\lambda(I(x, t)) + m_\lambda(I(x, |x - y|))} \frac{t|x - \tilde{x}|}{(t + |x - y|)^2}. \quad (2.8)$$

Using the mean value theorem and (1.7), we obtain that

$$\begin{aligned} F &\lesssim \int_0^\pi t^{-2\lambda-1} \left| \phi \left(\frac{\sqrt{x^2 + y^2 - 2xy \cos \theta}}{t} \right) - \phi \left(\frac{\sqrt{\tilde{x}^2 + y^2 - 2\tilde{x}y \cos \theta}}{t} \right) \right| (\sin \theta)^{2\lambda-1} d\theta \\ &\lesssim \int_0^\pi t^{-2\lambda-2} \left| \phi' \left(\frac{\sqrt{\xi^2 + y^2 - 2\xi y \cos \theta}}{t} \right) \right| |x - \tilde{x}| (\sin \theta)^{2\lambda-1} d\theta \\ &\lesssim \int_0^\pi \frac{t|x - \tilde{x}| (\sin \theta)^{2\lambda-1}}{(t^2 + \xi^2 + y^2 - 2\xi y \cos \theta)^{\lambda+\frac{3}{2}}} d\theta, \end{aligned} \quad (2.9)$$

where $\alpha \in (0, 1)$ and $\xi \equiv (1 - \alpha)x + \alpha\tilde{x}$.

We now prove (2.8) by considering the following two cases.

Case (i) $t \geq x$ or $|x - y| \geq x/2$. In this case, (2.8) follows from

$$F \lesssim \frac{t|x - \tilde{x}|}{(t + |x - y|)^{2\lambda+3}}. \quad (2.10)$$

By $|x - \tilde{x}| \leq (t + |x - y|)/2$ and the choice of ξ , we have

$$t + |\xi - y| = t + |(1 - \alpha)x + \alpha\tilde{x} - y| \geq t + |x - y| - \alpha|x - \tilde{x}| > (t + |x - y|)/2. \quad (2.11)$$

Then (2.10) follows from (2.11) together with (2.9) and (2.6) easily.

Case (ii) $t < x$ and $|x - y| < x/2$. In this case,

$$|x - \xi| < |x - \tilde{x}| \leq (t + |x - y|)/2 < 3x/4$$

and hence $y \sim x \sim \tilde{x} \sim \xi$. This fact together with the fact that for all $\theta \in (0, \pi/2]$, $\sin \theta \sim \theta$ and $1 - \cos \theta \geq 2(\theta/\pi)^2$ further implies that

$$\begin{aligned} F_1 &\equiv \int_0^{\pi/2} \frac{t|x - \tilde{x}| (\sin \theta)^{2\lambda-1}}{[t^2 + |\xi - y|^2 + 2\xi y(1 - \cos \theta)]^{\lambda+\frac{3}{2}}} d\theta \\ &\lesssim \int_0^{\pi/2} \frac{t|x - \tilde{x}| \theta^{2\lambda-1}}{[t^2 + |\xi - y|^2 + 4\xi y \theta^2/\pi^2]^{\lambda+\frac{3}{2}}} d\theta \\ &\lesssim \frac{t|x - \tilde{x}|}{(\xi y)^\lambda (t + |x - y|)^3} \int_0^\infty \frac{\beta^{2\lambda-1}}{(1 + \beta^2)^{\lambda+\frac{3}{2}}} d\beta \lesssim \frac{t|x - \tilde{x}|}{x^{2\lambda} (t + |x - y|)^3}. \end{aligned}$$

On the other hand, from (2.11) and (2.6), we deduce that

$$\begin{aligned} F_2 &\equiv \int_{\pi/2}^\pi \frac{t|x - \tilde{x}| (\sin \theta)^{2\lambda-1}}{(t^2 + \xi^2 + y^2 - 2\xi y \cos \theta)^{\lambda+\frac{3}{2}}} d\theta \\ &\lesssim \frac{t|x - \tilde{x}|}{x^{2\lambda} (t + |x - y|)^3} \int_{\pi/2}^\pi (\sin \theta)^{2\lambda-1} d\theta \lesssim \frac{t|x - \tilde{x}|}{x^{2\lambda} (t + |x - y|)^3}. \end{aligned}$$

Combining the estimates of F_1 and F_2 , we obtain $F \lesssim \frac{t|x-\tilde{x}|}{x^{2\lambda}(t+|x-y|)^3}$, which implies (2.8).

Similarly, to show that $\{\Phi_t\}_{t>0}$ satisfies Definition 1.1(iv), by (2.3), it suffices to show that for all $x, \tilde{x}, y, \tilde{y}, t \in (0, \infty)$ with $|x - \tilde{x}| \leq (t + |x - y|)/3$ and $|y - \tilde{y}| \leq (t + |x - y|)/3$,

$$\begin{aligned} G &\equiv |[\Phi_t(x, y) - \Phi_t(x, \tilde{y})] - [\Phi_t(\tilde{x}, y) - \Phi_t(\tilde{x}, \tilde{y})]| \\ &\lesssim \frac{1}{m_\lambda(I(x, t)) + m_\lambda(I(x, |x - y|))} \frac{t|x - \tilde{x}||y - \tilde{y}|}{(t + |x - y|)^3}. \end{aligned} \quad (2.12)$$

Using the mean value theorem, (1.7) and (1.8), we obtain that

$$\begin{aligned} G &\lesssim \int_0^\pi t^{-2\lambda-1} |x - \tilde{x}| |y - \tilde{y}| \left[\left| \frac{1}{t^2} \phi'' \left(\frac{\sqrt{\xi_1^2 + \xi_2^2 - 2\xi_1\xi_2 \cos \theta}}{t} \right) \right| \right. \\ &\quad \left. + \frac{1}{t \sqrt{\xi_1^2 + \xi_2^2 - 2\xi_1\xi_2 \cos \theta}} \left| \phi' \left(\frac{\sqrt{\xi_1^2 + \xi_2^2 - 2\xi_1\xi_2 \cos \theta}}{t} \right) \right| \right] (\sin \theta)^{2\lambda-1} d\theta \\ &\lesssim \int_0^\pi \frac{t|x - \tilde{x}||y - \tilde{y}|(\sin \theta)^{2\lambda-1}}{(t^2 + \xi_1^2 + \xi_2^2 - 2\xi_1\xi_2 \cos \theta)^{\lambda+2}} d\theta, \end{aligned}$$

where $\alpha, \beta \in (0, 1)$, $\xi_1 \equiv (1 - \alpha)x + \alpha\tilde{x}$ and $\xi_2 \equiv (1 - \beta)y + \beta\tilde{y}$.

To prove (2.12), we consider the following two cases.

Case (i) $t \geq x$ or $|x - y| \geq x/2$ or $t \geq y$. In this case, by $|x - \tilde{x}| \leq (t + |x - y|)/3$ and $|y - \tilde{y}| \leq (t + |x - y|)/3$, we have

$$t + |\xi_2 - \xi_1| \geq t + |y - x| - |x - \tilde{x}| - |y - \tilde{y}| \geq (t + |x - y|)/3. \quad (2.13)$$

This together with (2.6) yields that

$$G \lesssim \frac{t|x - \tilde{x}||y - \tilde{y}|}{(t + |x - y|)^{2\lambda+4}},$$

which implies (2.12) in this case.

Case (ii) $t < x$, $|x - y| < x/2$ and $t < y$. In this case, $\tilde{x}, y \in (x/2, 3x/2)$ and $\tilde{y} \in (x/6, 13x/6)$. Moreover, we see that $\xi_2 \sim \tilde{y} \sim y \sim x \sim \tilde{x} \sim \xi_1$. From this together with (2.13), (2.6) and the fact that for all $\theta \in (0, \pi/2]$, $\sin \theta \sim \theta$ and $1 - \cos \theta \geq 2(\theta/\pi)^2$, we further deduce that

$$\begin{aligned} G &\lesssim \int_0^{\pi/2} \frac{t|x - \tilde{x}||y - \tilde{y}|(\sin \theta)^{2\lambda-1}}{[t^2 + |\xi_1 - \xi_2|^2 + 2\xi_1\xi_2(1 - \cos \theta)]^{\lambda+2}} d\theta + \int_{\pi/2}^\pi \dots \\ &\lesssim \frac{t|x - \tilde{x}||y - \tilde{y}|}{(\xi_1\xi_2)^\lambda(t + |\xi_1 - \xi_2|)^4} \int_0^\infty \frac{\beta^{2\lambda-1}}{(1 + \beta^2)^{\lambda+2}} d\beta + \frac{t|x - \tilde{x}||y - \tilde{y}|}{(t^2 + \xi_1^2 + \xi_2^2)^{\lambda+2}} \\ &\lesssim \frac{t|x - \tilde{x}||y - \tilde{y}|}{x^{2\lambda}(t + |x - y|)^4}. \end{aligned}$$

This implies (2.12) and hence finishes the proof of Lemma 2.1. \square

Proof of Theorem 1.1. Let $p, \epsilon, \beta, \gamma$ and a be as in Theorem 1.1, $\phi \in Z^{[\lambda]}$ and $f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))'$. From [27, Theorem 3.1] (see also [11, Corollary 1.8]) together with Lemma

2.1, we deduce that $f \in H^p((0, \infty), dm_\lambda)$ if and only if $\Phi^+(f) \in L^p((0, \infty), dm_\lambda)$ and $\|f\|_{H^p((0, \infty), dm_\lambda)} \sim \|\Phi^+(f)\|_{L^p((0, \infty), dm_\lambda)}$. Furthermore, [10, Corollary 4.18] implies that $f \in H^p((0, \infty), dm_\lambda)$ if and only if $\Phi^*(f) \in L^p((0, \infty), dm_\lambda)$ if and only if $G^{(\epsilon, \beta, \gamma)}(f) \in L^p((0, \infty), dm_\lambda)$; moreover,

$$\|f\|_{H^p((0, \infty), dm_\lambda)} \sim \|\Phi^*(f)\|_{L^p((0, \infty), dm_\lambda)} \sim \|G^{(\epsilon, \beta, \gamma)}(f)\|_{L^p((0, \infty), dm_\lambda)}.$$

Finally, it follows from [14, Theorems 5.13 and 5.16] that $f \in H^p((0, \infty), dm_\lambda)$ if and only if $g(f) \in L^p((0, \infty), dm_\lambda)$ if and only if $S_a(f) \in L^p((0, \infty), dm_\lambda)$; moreover,

$$\|f\|_{H^p((0, \infty), dm_\lambda)} \sim \|g(f)\|_{L^p((0, \infty), dm_\lambda)} \sim \|S_a(f)\|_{L^p((0, \infty), dm_\lambda)}.$$

Combining these facts, we then complete the proof of Theorem 1.1. \square

3. Proof of Theorem 1.2

In this section, we present the proof of Theorem 1.2. We begin with the following lemma on the boundedness of $\tilde{R}_{\Delta_\lambda}$.

Lemma 3.1. *Let $\epsilon \in (0, 1]$ and $\beta, \gamma, \tilde{\epsilon} \in (0, \epsilon)$. Then $\tilde{R}_{\Delta_\lambda}$ is bounded from $\tilde{\mathcal{G}}_0^{\tilde{\epsilon}}(\beta, \gamma)$ to $\mathcal{G}_0^{\tilde{\epsilon}}(\beta, \gamma)$.*

Proof. By [19, p. 87], we have that R_{Δ_λ} is bounded on $L^r((0, \infty), dm_\lambda)$ for $r \in (1, \infty)$, and so is $\tilde{R}_{\Delta_\lambda}$. Recall that the kernel, denoted by $\tilde{R}_{\Delta_\lambda}(x, y)$, of $\tilde{R}_{\Delta_\lambda}$ satisfies that for all $x, y \in (0, \infty)$,

$$\tilde{R}_{\Delta_\lambda}(x, y) = Q_0^{[\lambda]}(y, x) \equiv -\frac{2\lambda}{\pi} \int_0^\pi \frac{(y - x \cos \theta)(\sin \theta)^{2\lambda-1}}{(x^2 + y^2 - 2xy \cos \theta)^{\lambda+1}} d\theta;$$

see Lemma 2.42 in [3]. It is easy to see that $\tilde{R}_{\Delta_\lambda}(x, y)$ satisfies Conditions (i)-(iii) of Theorem 2.18 in [14] with $\epsilon = 1$ therein (see also [5]), which means the $\tilde{R}_{\Delta_\lambda}(x, y)$ also satisfies the hypotheses of Corollary 2.24 in [14]. By this and [14, Remark 2.14(iii), Remark 2.17, Corollary 2.24], Lemma 3.1 is reduced to showing that $\tilde{R}_{\Delta_\lambda}(1) \in \text{BMO}((0, \infty), dm_\lambda)$ is a constant, where $f \in L_{\text{loc}}^1(0, \infty)$ is called to belong to the space $\text{BMO}((0, \infty), dm_\lambda)$ if

$$\sup_{x, r \in (0, \infty)} \frac{1}{m_\lambda(I(x, r))} \int_{I(x, r)} \left| f(y) - \frac{1}{m_\lambda(I(x, r))} \int_{I(x, r)} f(z) dm_\lambda(z) \right| dm_\lambda(y) < \infty.$$

Recall that $\text{BMO}((0, \infty), dm_\lambda)$ is the dual space of $H^1((0, \infty), dm_\lambda)$ and R_{Δ_λ} is bounded from $H^1((0, \infty), dm_\lambda)$ to $L^1((0, \infty), dm_\lambda)$ (see [7]). Then by Theorem 4.10 in [20], we see that $\tilde{R}_{\Delta_\lambda}$ is bounded from $L^\infty((0, \infty), dm_\lambda)$ to $\text{BMO}((0, \infty), dm_\lambda)$, which implies that $\tilde{R}_{\Delta_\lambda}(1) \in \text{BMO}((0, \infty), dm_\lambda)$. Moreover, for any $x \in (0, \infty)$,

$$\begin{aligned} \tilde{R}_{\Delta_\lambda}(1)(x) &= \lim_{\delta \rightarrow 0} \int_{0, |x-y| > \delta}^\infty Q_0^{[\lambda]}(y, x) y^{2\lambda} dy \\ &= -\frac{2\lambda}{\pi} \lim_{\delta \rightarrow 0} \int_{0, |x-y| > \delta}^\infty \int_0^\pi \frac{(y - x \cos \theta)(\sin \theta)^{2\lambda-1}}{(x^2 + y^2 - 2xy \cos \theta)^{\lambda+1}} d\theta y^{2\lambda} dy \\ &= -\frac{2\lambda}{\pi} \lim_{\delta \rightarrow 0} \int_{0, |1-z| > \delta}^\infty \int_0^\pi \frac{(z - \cos \theta)(\sin \theta)^{2\lambda-1}}{(z^2 + 1 - 2z \cos \theta)^{\lambda+1}} d\theta z^{2\lambda} dz = \tilde{R}_{\Delta_\lambda}(1)(1). \end{aligned}$$

This implies that $\tilde{R}_{\Delta_\lambda}(1)$ is a constant and hence finishes the proof Lemma 3.1. \square

We next establish the boundedness of R_{Δ_λ} on $H^p((0, \infty), dm_\lambda)$. To this end, for any $x_0, x, r \in (0, \infty)$, let $d_\lambda(x, x_0) \equiv |\int_{x_0}^x y^{2\lambda} dy|$, $I_{d_\lambda}(x_0, r) \equiv \{x \in (0, \infty) : d_\lambda(x, x_0) < r\}$ and for all $k \in \mathbb{N}$,

$$R_k(I_{d_\lambda}(x_0, r)) \equiv \{x \in (0, \infty) : 2^{k-1}m_\lambda(I_{d_\lambda}(x_0, r)) \leq d_\lambda(x, x_0) < 2^k m_\lambda(I_{d_\lambda}(x_0, r))\}.$$

Then d_λ is the measure distance and satisfies that for any $x_0, r \in (0, \infty)$,

$$m_\lambda(I_{d_\lambda}(x_0, r)) \sim r; \quad (3.1)$$

see Theorem 3 in [17]. Write $H^p((0, \infty), dm_\lambda)$ as $H^p((0, \infty), \rho, dm_\lambda)$ for the moment, where $\rho(x, y) \equiv |x - y|$ for all $x, y \in (0, \infty)$. If we replace ρ by d_λ in Definition 1.3, we then obtain the corresponding $H^p((0, \infty), d_\lambda, dm_\lambda)$ -atoms and the Hardy spaces $H^p((0, \infty), d_\lambda, dm_\lambda)$. In [15, Theorems 2.1], it was proved that for any $p \in ((2\lambda + 1)/(2\lambda + 2), 1]$, the spaces

$$H^p((0, \infty), \rho, dm_\lambda) = H^p((0, \infty), d_\lambda, dm_\lambda) \quad (3.2)$$

with equivalent norms.

We recall the notion of molecules in [15] as follows; see also [7, 18].

Definition 3.1. Let $p \in ((2\lambda + 1)/(2\lambda + 2), 1]$ and $\eta \equiv \{\eta_k\}_{k \in \mathbb{N}} \subset [0, \infty)$ such that $\sum_{k=1}^\infty k\eta_k < \infty$ when $p = 1$, or $\sum_{k=1}^\infty (\eta_k)^p 2^{k(1-p)} < \infty$ when $p \in (0, 1)$. A function $M \in L^2((0, \infty), dm_\lambda)$ is called a $(p, 2, \eta)$ -molecule centered at an interval $I_{d_\lambda} \equiv I_{d_\lambda}(y_0, r_0)$ for some $y_0, r_0 \in (0, \infty)$, if

- (M)_i $\|M\|_{L^2((0, \infty), dm_\lambda)} \leq [m_\lambda(I_{d_\lambda})]^{1/2-1/p}$;
- (M)_{ii} for all $k \in \mathbb{N}$, $\|M\chi_{R_k(I_{d_\lambda})}\|_{L^2((0, \infty), dm_\lambda)} \leq \eta_k 2^{-k/2} [m_\lambda(I_{d_\lambda})]^{1/2-1/p}$;
- (M)_{iii} $\int_0^\infty M(x)x^{2\lambda} dx = 0$.

Lemma 3.2. Let $p \in ((2\lambda + 1)/(2\lambda + 2), 1]$. Then R_{Δ_λ} is bounded from $H^p((0, \infty), dm_\lambda)$ to $L^p((0, \infty), dm_\lambda)$ and bounded on $H^p((0, \infty), dm_\lambda)$.

Proof. We only show that R_{Δ_λ} is bounded on $H^p((0, \infty), dm_\lambda)$, since the proof for the boundedness of R_{Δ_λ} from $H^p((0, \infty), dm_\lambda)$ to $L^p((0, \infty), dm_\lambda)$ is similar and easier. Assume that a is an $H^p((0, \infty), dm_\lambda)$ -atom such that $\text{supp}(a) \subset I \equiv I(x_0, r)$ for some $x_0, r \in (0, \infty)$. By Theorem 1.1 in [26] (see also [10, Theorem 5.9]), we only need to show that there exists a positive constant C , independent of a , such that $\|R_{\Delta_\lambda}(a)\|_{H^p((0, \infty), dm_\lambda)} \leq C$. From Theorem 2.2 in [15], we deduce that there exists a positive constant \tilde{C} such that for any $(p, 2, \eta)$ -molecule M as in Definition 3.1, $M \in H^p((0, \infty), d_\lambda, dm_\lambda)$ and $\|M\|_{H^p((0, \infty), d_\lambda, dm_\lambda)} \leq \tilde{C}$. Via this and (3.2), it suffices to show that $R_{\Delta_\lambda}(a)$ is a $(p, 2, \eta)$ -molecule centered at the interval $I_{d_\lambda} \equiv I_{d_\lambda}(x_0, m_\lambda(I))$ with $\eta \equiv \{2^{-\frac{k}{2\lambda+1}}\}_{k \in \mathbb{N}}$.

Recall that R_{Δ_λ} is bounded from $H^1((0, \infty), dm_\lambda)$ to $L^1((0, \infty), dm_\lambda)$. This together with $a \in H^1((0, \infty), dm_\lambda)$ implies that $R_{\Delta_\lambda}(a) \in L^1((0, \infty), dm_\lambda)$. Then from this observation and Lemma 3.1, it follows that

$$\int_0^\infty R_{\Delta_\lambda}(a)(x)x^{2\lambda} dx = \langle R_{\Delta_\lambda}(a), 1 \rangle = \langle a, \tilde{R}_{\Delta_\lambda}(1) \rangle = 0.$$

On the other hand, by (3.1), we see that

$$m_\lambda(I_{d_\lambda}) \sim m_\lambda(I) \quad (3.3)$$

and for each $k \in \mathbb{N}$,

$$m_\lambda(R_k(I_{d_\lambda})) \lesssim 2^k m_\lambda(I). \quad (3.4)$$

Applying the boundedness of R_{Δ_λ} on $L^2((0, \infty), dm_\lambda)$, Definition 1.3 and (3.3), we have

$$\|R_{\Delta_\lambda}(a)\|_{L^2((0, \infty), dm_\lambda)} \lesssim \|a\|_{L^2((0, \infty), dm_\lambda)} \lesssim [m_\lambda(I)]^{1/2-1/p} \lesssim [m_\lambda(I_{d_\lambda})]^{1/2-1/p}. \quad (3.5)$$

Thus, via (3.3), Lemma 3.2 is reduced to showing that for each $k \in \mathbb{N}$,

$$\left\| [R_{\Delta_\lambda}(a)] \chi_{R_k(I_{d_\lambda})} \right\|_{L^2((0, \infty), dm_\lambda)} \lesssim (2^k)^{-\frac{\lambda+3/2}{2\lambda+1}} [m_\lambda(I)]^{1/2-1/p}. \quad (3.6)$$

If $x \in 2I$, then by (1.5), we see that

$$d_\lambda(x, x_0) = \left| \int_x^{x_0} y^{2\lambda} dy \right| \leq m_\lambda(2I) \lesssim 2^{2\lambda+1} m_\lambda(I).$$

This together with (3.1) and (3.3) implies that there exists $K_0 \in \mathbb{N}$ such that $(2I) \cap R_k(I_{d_\lambda}) = \emptyset$ for all $k > K_0$. Furthermore, (3.5) implies (3.6) for all $k \in \mathbb{N}$ and $k \leq K_0$.

To prove (3.6) for $k > K_0$, we first claim that for any $x \in (0, \infty) \setminus (2I)$,

$$|R_{\Delta_\lambda}(a)(x)| \lesssim \frac{r[m_\lambda(I)]^{1-1/p}}{|x - x_0|^{2\lambda+2}}, \quad (3.7)$$

and when $x_0 \geq 2r$,

$$|R_{\Delta_\lambda}(a)(x)| \lesssim \frac{r[m_\lambda(I)]^{1-1/p}}{|x - x_0|^2 x^\lambda x_0^\lambda}. \quad (3.8)$$

Indeed, by the vanishing moment of a and the mean value theorem, we obtain that

$$\begin{aligned} |R_{\Delta_\lambda}(a)(x)| &\sim \left| \int_I \left[Q_0^{[\lambda]}(x, y) - Q_0^{[\lambda]}(x, x_0) \right] a(y) y^{2\lambda} dy \right| \\ &\lesssim r \int_I \int_0^\pi \frac{(\sin \theta)^{2\lambda-1}}{(x^2 + \xi^2 - 2x\xi \cos \theta)^{\lambda+1}} d\theta |a(y)| y^{2\lambda} dy, \end{aligned} \quad (3.9)$$

where $\xi \equiv \alpha x_0 + (1 - \alpha)y$ for $y \in I$ and some $\alpha \in (0, 1)$.

On the one hand, since $x \in (0, \infty) \setminus (2I)$, then $x^2 + \xi^2 - 2x\xi \cos \theta \geq (x - \xi)^2 \gtrsim (x - x_0)^2$, which together with (3.9) implies (3.7). On the other hand, when $x_0 \geq 2r$, we have that for all $y \in I = (x_0 - r, x_0 + r)$, $y \sim x_0$. This implies that $\xi \sim x_0$. Then by some estimates similar to those used in the estimate (2.7), we further see that

$$\int_0^\pi \frac{(\sin \theta)^{2\lambda-1}}{(x^2 + \xi^2 - 2x\xi \cos \theta)^{\lambda+1}} d\theta \lesssim \frac{1}{|x - x_0|^2 x^\lambda x_0^\lambda}.$$

This together with (3.9) and $\|a\|_{L^1((0, \infty), dm_\lambda)} \leq [m_\lambda(I)]^{1-1/p}$ implies (3.8).

For $x > 2x_0$, the mean value theorem implies that

$$d_\lambda(x, x_0) \sim x^{2\lambda+1} - x_0^{2\lambda+1} \sim \zeta^{2\lambda}(x - x_0) \lesssim x^{2\lambda}(x - x_0) \lesssim (x - x_0)^{2\lambda+1}$$

for some $\beta \in (0, 1)$ and $\zeta \equiv \beta x_0 + (1 - \beta)x$. This together with $x \in R_k(I_{d_\lambda})$ leads to that $2^k m_\lambda(I) \lesssim (x - x_0)^{2\lambda+1}$. By this, (3.7), (3.4) and (1.4), we further see that

$$\begin{aligned} F_1 &\equiv \left\{ \int_{2x_0}^{\infty} [R_{\Delta_\lambda}(a)(x)]^2 \chi_{R_k(I_{d_\lambda})}(x) dm_\lambda(x) \right\}^{1/2} \\ &\lesssim r[m_\lambda(I)]^{1-1/p} \left\{ \int_{R_k(I_{d_\lambda})} \frac{1}{|x - x_0|^{4\lambda+4}} dm_\lambda(x) \right\}^{1/2} \\ &\lesssim r[m_\lambda(I)]^{1-1/p} [2^k m_\lambda(I)]^{-\frac{\lambda+3/2}{2\lambda+1}} \lesssim [m_\lambda(I)]^{1/2-1/p} (2^k)^{-\frac{\lambda+3/2}{2\lambda+1}}. \end{aligned}$$

Similarly,

$$F_2 \equiv \left\{ \int_0^{x_0/2} [R_{\Delta_\lambda}(a)(x)]^2 \chi_{R_k(I_{d_\lambda})}(x) dm_\lambda(x) \right\}^{1/2} \lesssim [m_\lambda(I)]^{1/2-1/p} (2^k)^{-\frac{\lambda+3/2}{2\lambda+1}}.$$

Assume that there exists $x \in [x_0/2, 2x_0] \cap R_k(I_{d_\lambda})$. Then since $R_k(I_{d_\lambda}) \cap (2I) = \emptyset$, we see that $[x_0/2, 2x_0] \not\subseteq (2I)$, which further implies that $x_0 > 2r$. Using this fact together with the mean value theorem and the fact that $x \sim x_0$, we have that $d_\lambda(x, x_0) \sim x_0^{2\lambda}|x - x_0|$. This implies that for any $x \in R_k(I_{d_\lambda})$, $|x - x_0| \sim 2^k m_\lambda(I) x_0^{-2\lambda}$. Thus, from this fact together with (3.8) and (1.4), we deduce that

$$\begin{aligned} F_3 &\equiv \left\{ \int_{x_0/2}^{2x_0} [R_{\Delta_\lambda}(a)(x)]^2 \chi_{R_k(I_{d_\lambda})}(x) dm_\lambda(x) \right\}^{1/2} \\ &\lesssim r[m_\lambda(I)]^{1-1/p} \left\{ \int_{R_k(I_{d_\lambda})} \frac{1}{|x - x_0|^{4\lambda} x_0^{4\lambda}} dm_\lambda(x) \right\}^{1/2} \\ &\lesssim r[m_\lambda(I)]^{1-1/p} \left\{ \frac{m_\lambda(R_k(I_{d_\lambda}))}{[2^k m_\lambda(I)]^4 x_0^{-4\lambda}} \right\}^{1/2} \lesssim r[m_\lambda(I)]^{1-1/p} [2^k m_\lambda(I)]^{-3/2} x_0^{2\lambda} \\ &\lesssim [m_\lambda(I)]^{1/2-1/p} (2^k)^{-\frac{3}{2}} \lesssim [m_\lambda(I)]^{1/2-1/p} (2^k)^{-\frac{\lambda+3/2}{2\lambda+1}}. \end{aligned}$$

Combining the estimates of F_i for $i \in \{1, 2, 3\}$, we obtain (3.6), which completes the proof of Lemma 3.2. \square

Recall that for any $t, x, y \in (0, \infty)$, the kernel $P_t^{[\lambda]}(x, y)$ of $P_t^{[\lambda]}$ satisfies that

$$\begin{aligned} P_t^{[\lambda]}(x, y) &= \int_0^\infty e^{-tz} (xz)^{-\lambda+1/2} J_{\lambda-1/2}(xz) (yz)^{-\lambda+1/2} J_{\lambda-1/2}(yz) dm_\lambda(z) \\ &= \frac{2\lambda t}{\pi} \int_0^\pi \frac{(\sin \theta)^{2\lambda-1}}{(x^2 + y^2 + t^2 - 2xy \cos \theta)^{\lambda+1}} d\theta; \end{aligned} \tag{3.10}$$

see [3] or [25]. Moreover, $\{P_t^{[\lambda]}\}_{t>0}$ is a contraction semigroup on $L^r((0, \infty), dm_\lambda)$ for all $r \in [1, \infty]$. Also, for any $f \in L^r((0, \infty), dm_\lambda)$ with $r \in [1, \infty]$, the Poisson integral $u(t, x) \equiv P_t^{[\lambda]}(f)(x)$ satisfies the differential equation that for all $t, x \in (0, \infty)$,

$$\partial_t^2 u(t, x) + \partial_x^2 u(t, x) + \frac{2\lambda}{x} \partial_x u(t, x) = 0. \quad (3.11)$$

Moreover, let $v(t, x) \equiv Q_t^{[\lambda]}(f)(x)$. Then u and v satisfy the following Cauchy-Riemann type equations:

$$\partial_t v + \partial_x u = 0, \quad \partial_t v - \partial_x u - \frac{2\lambda}{x} v = 0; \quad (3.12)$$

see [19, 3].

The following lemma was proved in [19].

Lemma 3.3. *Suppose that*

- (i) $u(t, x)$ is continuous in $t \in [0, \infty)$, $x \in \mathbb{R}$ and even in x ;
- (ii) In the region where $u(t, x) > 0$, u is of class C^2 and satisfies $\partial^2 u / \partial t^2 + \partial^2 u / \partial x^2 + 2\lambda x^{-1} \partial u / \partial x \geq 0$ there;
- (iii) $u(0, x) = 0$;
- (iv) For some $r \in [1, \infty)$, there exists a positive constant \tilde{C} such that

$$\sup_{0 < t < \infty} \int_0^\infty |u(t, x)|^r dm_\lambda(x) \leq \tilde{C} < \infty. \quad (3.13)$$

Then $u(t, x) \leq 0$.

For any function u on $(0, \infty) \times \mathbb{R}$ and $x \in (0, \infty)$, let $u^*(x) \equiv \sup_{t>0} |u(t, x)|$. We then have a variant of Lemma 2.38 in [3] as follows.

Lemma 3.4. *Let $p \in ((2\lambda)/(2\lambda + 1), 1]$. Assume that $u(t, x)$ and $v(t, x)$ are, respectively, even and odd (with respect to x) real valued C^2 functions on $(0, \infty) \times \mathbb{R}$ and satisfy the following Cauchy-Riemann type equation (3.12). Let $F \equiv (u^2 + v^2)^{1/2}$ and suppose that*

$$\sup_{t>0} \int_0^\infty [F(t, x)]^p dm_\lambda(x) < \infty. \quad (3.14)$$

Then $u^* \in L^p((0, \infty), dm_\lambda)$ and there exists a positive constant C , depending only on λ and p , such that

$$\|u^*\|_{L^p((0, \infty), dm_\lambda)} \leq C \sup_{t>0} \int_0^\infty [F(t, x)]^p dm_\lambda(x).$$

Proof. By [19, Lemma 5], for any $p \geq (2\lambda)/(2\lambda + 1)$, we have that for all $(t, x) \in (0, \infty) \times \mathbb{R}$ such that $F(t, x) > 0$,

$$\frac{\partial^2 [F(t, x)]^p}{\partial x^2} + \frac{\partial^2 [F(t, x)]^p}{\partial t^2} + \frac{2\lambda}{x} \frac{\partial [F(t, x)]^p}{\partial x} \geq 0. \quad (3.15)$$

Let $p \in ((2\lambda)/(2\lambda + 1), 1]$ and $F_\delta(x) \equiv F(\delta, x)$ for any $(\delta, x) \in (0, \infty) \times \mathbb{R}$. Take $q \in ((2\lambda)/(2\lambda + 1), p)$ and $r \equiv p/q$. An application of (3.14) leads to that for all $\delta \in (0, \infty)$, $F_\delta^q \in L^r((0, \infty), dm_\lambda)$.

We claim that for any $t, \delta, x \in (0, \infty)$,

$$[F(t + \delta, x)]^q \leq P_t^{[\lambda]}(F_\delta^q)(x). \quad (3.16)$$

To see this, let $P_0^{[\lambda]}(F_\delta^q) \equiv \lim_{t \rightarrow 0+} P_t^{[\lambda]}(F_\delta^q)$ and define $V_\delta(t, x)$ by setting, for all $t \in [0, \infty)$ and $x \in \mathbb{R}$,

$$V_\delta(t, x) \equiv [F(t + \delta, x)]^q - \tilde{P}_t^{[\lambda]}(F_\delta^q)(x),$$

where $\tilde{P}_t^{[\lambda]}(F_\delta^q)$ is the odd extension to \mathbb{R} of $P_t^{[\lambda]}(F_\delta^q)$. To show (3.16), it suffices to show that Lemma 3.3 holds for V_δ . In fact, it is obvious that V_δ is continuous on $[0, \infty) \times \mathbb{R}$. Since F is nonnegative, $P_t^{[\lambda]}(F_\delta^q)$ is also nonnegative. Thus, if $(t, x) \in (0, \infty)$ such that $V_\delta(t, x) > 0$, then $[F(t + \delta, x)]^q > 0$, which together with (3.11) and (3.15) implies that

$$\frac{\partial^2 V_\delta(t, x)}{\partial x^2} + \frac{\partial^2 V_\delta(t, x)}{\partial t^2} + \frac{2\lambda}{x} \frac{\partial V_\delta(t, x)}{\partial x} \geq 0$$

and hence Lemma 3.3(ii). Moreover, by the continuity of F and the fact that $P_t^{[\lambda]}f \rightarrow f$ in $L^p((0, \infty), dm_\lambda)$ for any $f \in L^p((0, \infty), dm_\lambda)$ with $p \in [1, \infty)$, we see that $P_0^{[\lambda]}(F_\delta^q) = F_\delta^q$, which yields that for any $x \in \mathbb{R}$, $V_\delta(0, x) = 0$. Thus, Lemma 3.3(iii) holds.

Finally, by the uniform boundedness of $P_t^{[\lambda]}$ on $L^r((0, \infty), dm_\lambda)$ for $r \in [1, \infty]$ and (3.14), we have

$$\begin{aligned} & \sup_{0 < t < \infty} \int_0^\infty |V_\delta(t, x)|^r dm_\lambda(x) \\ & \lesssim \sup_{0 < t < \infty} \int_0^\infty [F(t, x)]^p dm_\lambda(x) + \sup_{0 < t < \infty} \int_0^\infty [P_t^{[\lambda]}(F_\delta^q)(x)]^r dm_\lambda(x) \\ & \lesssim \sup_{t > 0} \int_0^\infty [F(t, x)]^p dm_\lambda(x) < \infty. \end{aligned}$$

Therefore, (3.13) holds for V_δ and consequently, the claim follows from Lemma 3.3.

Since $\{F_\delta^q\}_{\delta > 0}$ is bounded on $L^r((0, \infty), dm_\lambda)$, and $L^r((0, \infty), dm_\lambda)$ is reflexive, there exists a sequence $\delta_k \downarrow 0$ and $h \in L^r((0, \infty), dm_\lambda)$ such that $F_{\delta_k}^q$ converges weakly to h in $L^r((0, \infty), dm_\lambda)$ as $k \rightarrow \infty$. Moreover, by the Hölder inequality, we see that

$$\begin{aligned} \|h\|_{L^r((0, \infty), dm_\lambda)}^r &= \left\{ \sup_{\|g\|_{L^{r'}((0, \infty), dm_\lambda)} \leq 1} \left| \int_0^\infty g(x) h(x) dm_\lambda(x) \right| \right\}^r \\ &= \left\{ \sup_{\|g\|_{L^{r'}((0, \infty), dm_\lambda)} \leq 1} \lim_{k \rightarrow \infty} \left| \int_0^\infty g(x) [F_{\delta_k}(x)]^q dm_\lambda(x) \right| \right\}^r \\ &\leq \limsup_{k \rightarrow \infty} \left\| F_{\delta_k}^q \right\|_{L^r((0, \infty), dm_\lambda)}^r \leq \sup_{t > 0} \int_0^\infty [F(t, x)]^p dm_\lambda(x). \end{aligned} \quad (3.17)$$

Since F is continuous, then for any $x \in (0, \infty)$, $[F(t + \delta_k, x)]^q \rightarrow [F(t, x)]^q$ as $k \rightarrow \infty$. Observe that for any fixed $x \in (0, \infty)$, $P_t^{[\lambda]}(x, \cdot) \in L^{r'}((0, \infty), dm_\lambda)$. From this fact, we deduce that for each $x \in (0, \infty)$, $P_t^{[\lambda]}(F_{\delta_k}^q)(x) \rightarrow P_t^{[\lambda]}(h)(x)$ as $k \rightarrow \infty$. Thus, by these facts and (3.16), we have that for any $t, x \in (0, \infty)$,

$$[F(t, x)]^q = \lim_{k \rightarrow \infty} [F(t + \delta_k, x)]^q \leq \lim_{k \rightarrow \infty} P_t^{[\lambda]}(F_{\delta_k}^q)(x) = P_t^{[\lambda]}(h)(x).$$

Therefore,

$$[u^*(x)]^q = \left[\sup_{0 < t < \infty} u(t, x) \right]^q \leq \sup_{0 < t < \infty} [F(t, x)]^q \leq \sup_{0 < t < \infty} P_t^{[\lambda]}(h)(x).$$

By this together with (c) in [19, p. 86] and (3.17), we then have

$$\begin{aligned} \|u^*\|_{L^p((0, \infty), dm_\lambda)}^p &= \|[u^*]^q\|_{L^r((0, \infty), dm_\lambda)}^r \lesssim \left\| \sup_{0 < t < \infty} P_t^{[\lambda]}(h) \right\|_{L^r((0, \infty), dm_\lambda)}^r \\ &\lesssim \|h\|_{L^r((0, \infty), dm_\lambda)}^r \lesssim \sup_{t > 0} \int_0^\infty [F(t, x)]^p dm_\lambda(x), \end{aligned}$$

which completes the proof of Lemma 3.4. \square

Proof of Theorem 1.2. We first assume that $f \in H^p((0, \infty), dm_\lambda)$. By Theorem 1.1, we have that $\sup_{\delta > 0} |f \#_\lambda \phi_\delta| \in L^p((0, \infty), dm_\lambda)$ and for all $\delta \in (0, \infty)$,

$$\|f \#_\lambda \phi_\delta\|_{L^p((0, \infty), dm_\lambda)} \lesssim \|f\|_{H^p((0, \infty), dm_\lambda)}.$$

Thus, (1.10) is reduced to showing that for all $\delta \in (0, \infty)$,

$$\|R_{\Delta_\lambda}(f \#_\lambda \phi_\delta)\|_{L^p((0, \infty), dm_\lambda)} \lesssim \|f\|_{H^p((0, \infty), dm_\lambda)}. \quad (3.18)$$

To this end, for each $\delta \in (0, \infty)$, let $\Phi_\delta(f) \equiv f \#_\lambda \phi_\delta$. By an argument similar to that used in the estimates of (3.5) and (3.6), we see that for any $H^p((0, \infty), dm_\lambda)$ -atom a , $\Phi_\delta(a)$ satisfies that for some interval I_{d_λ} ,

$$\|\Phi_\delta(a)\|_{L^2((0, \infty), dm_\lambda)} \lesssim [m_\lambda(I_{d_\lambda})]^{1/2-1/p},$$

and for each $k \in \mathbb{N}$,

$$\left\| [\Phi_\delta(a)] \chi_{R_k(I_{d_\lambda})} \right\|_{L^2((0, \infty), dm_\lambda)} \lesssim (2^k)^{-\frac{\lambda+3/2}{2\lambda+1}} [m_\lambda(I_{d_\lambda})]^{1/2-1/p}.$$

Observe that (2.2) and the Fubini theorem imply that $\int_0^\infty \Phi_\delta(a)(x) dm_\lambda(x) = 0$. These facts further yield that there exists a positive constant C , independent of a and δ , such that $\Phi_\delta(a)/C$ is a $(p, 2, \eta)$ -molecule with $\eta \equiv \{2^{-\frac{k}{2\lambda+1}}\}_{k \in \mathbb{N}}$ as in Definition 3.1. Moreover, by this together with [15, Theorem 2.2], we see that $\Phi_\delta(a) \in H^p((0, \infty), d_\lambda, dm_\lambda)$ and $\|\Phi_\delta(a)\|_{H^p((0, \infty), d_\lambda, dm_\lambda)} \lesssim 1$. This combined with [15, Theorem 2.1] further implies that $\Phi_\delta(a) \in H^p((0, \infty), dm_\lambda)$ and $\|\Phi_\delta(a)\|_{H^p((0, \infty), dm_\lambda)} \lesssim 1$. From this and [26, Theorem 1.1],

we deduce that $\{\Phi_\delta\}_{\delta>0}$ is bounded on $H^p((0, \infty), dm_\lambda)$ uniformly on $\delta \in (0, \infty)$. On the other hand, by Lemma 3.2, we obtain that R_{Δ_λ} is bounded from $H^p((0, \infty), dm_\lambda)$ to $L^p((0, \infty), dm_\lambda)$ for all $p \in ((2\lambda + 1)/(2\lambda + 2), 1]$. Combining these two facts, we obtain (3.18).

We now assume that (1.10) holds. For $\delta, t, x \in (0, \infty)$, let $u(t, x) \equiv P_t^{[\lambda]}(f\sharp_\lambda\phi_\delta)(x)$ and $v(t, x) \equiv Q_t^{[\lambda]}(f\sharp_\lambda\phi_\delta)(x)$. Moreover, define $F_\delta \equiv (\tilde{u}^2 + \tilde{v}^2)^{1/2}$, where \tilde{u} and \tilde{v} are even extension and odd extension of u and v with respect to x to \mathbb{R} , respectively. Then by (3.10), (1.9) and (3.12) together with the definitions of \tilde{u} and \tilde{v} , we have that \tilde{u}, \tilde{v} are C^2 functions on $(0, \infty) \times \mathbb{R}$ satisfying the Cauchy-Riemann type equations (3.12).

We prove that for all $x \in (0, \infty)$,

$$[F_\delta(t, x)]^p \leq P_t^{[\lambda]}([F_\delta(0, \cdot)]^p)(x). \quad (3.19)$$

To see this, for all $(t, x) \in [0, \infty) \times \mathbb{R}$, define

$$V_\delta(t, x) \equiv [F_\delta(t, x)]^p - \tilde{P}_t^{[\lambda]}([F_\delta(0, \cdot)]^p)(x),$$

where $\tilde{P}_t^{[\lambda]}([F_\delta(0, \cdot)]^p)$ is the even extension to \mathbb{R} of $P_t^{[\lambda]}([F_\delta(0, \cdot)]^p)$. As in the proof of Lemma 3.4, (3.19) is reduced to showing that V_δ satisfies Conditions (i)-(iv) of Lemma 3.3. Observe that V_δ is continuous on $[0, \infty) \times \mathbb{R}$. Moreover, we see that V_δ satisfies (ii) and (iii) of Lemma 3.3. It remains to show that V_δ satisfies (3.13). Since the assumption that f is restricted at infinity implies that $f\sharp_\lambda\phi_\delta \in L^r((0, \infty), dm_\lambda)$ for all $\delta \in (0, \infty)$ and $r \in [p, \infty]$, by the uniform boundedness of $P_t^{[\lambda]}$ and $Q_t^{[\lambda]}$ on $L^2((0, \infty), dm_\lambda)$ (see [19, p. 87]), we further obtain that

$$\begin{aligned} \int_0^\infty [F_\delta(t, x)]^2 dm_\lambda(x) &= \int_0^\infty \left[\left| P_t^{[\lambda]}(f\sharp_\lambda\phi_\delta)(x) \right|^2 + \left| Q_t^{[\lambda]}(f\sharp_\lambda\phi_\delta)(x) \right|^2 \right] dm_\lambda(x) \\ &\lesssim \int_0^\infty [f\sharp_\lambda\phi_\delta(x)]^2 dm_\lambda(x) < \infty. \end{aligned} \quad (3.20)$$

Observe that for almost every $x \in (0, \infty)$,

$$[F_\delta(0, x)]^2 = |f\sharp_\lambda\phi_\delta(x)|^2 + |R_{\Delta_\lambda}(f\sharp_\lambda\phi_\delta)(x)|^2.$$

This fact together with the boundedness of $P_t^{[\lambda]}$ on $L^{2/p}((0, \infty), dm_\lambda)$ and the boundedness of R_{Δ_λ} on $L^2((0, \infty), dm_\lambda)$ implies that

$$\begin{aligned} \int_0^\infty \left[P_t^{[\lambda]}(F_\delta^p(0, \cdot))(x) \right]^{2/p} dm_\lambda(x) &\lesssim \int_0^\infty |F_\delta(0, x)|^2 dm_\lambda(x) \\ &\lesssim \int_0^\infty \left[|f\sharp_\lambda\phi_\delta(x)|^2 + |R_{\Delta_\lambda}(f\sharp_\lambda\phi_\delta)(x)|^2 \right] dm_\lambda(x) \\ &\lesssim \int_0^\infty |f\sharp_\lambda\phi_\delta(x)|^2 dm_\lambda(x), \end{aligned}$$

from which and (3.20), we further deduce that

$$\begin{aligned} & \sup_{0 < t < \infty} \int_0^\infty |V_\delta(t, x)|^{2/p} dm_\lambda(x) \\ & \lesssim \sup_{0 < t < \infty} \int_0^\infty \left\{ [F_\delta(t, x)]^2 + \left[P_t^{[\lambda]}(F_\delta^p(0, \cdot))(x) \right]^{2/p} \right\} dm_\lambda(x) \\ & \lesssim \int_0^\infty |f \#_\lambda \phi_\delta(x)|^2 dm_\lambda(x) < \infty. \end{aligned}$$

Therefore, V_δ satisfies (3.13) and hence (3.19) follows from Lemma 3.3 immediately.

Using (3.19), (1.10) and the uniform boundedness of $\{P_t^{[\lambda]}\}_{t>0}$ on $L^1((0, \infty), dm_\lambda)$, we see that

$$\begin{aligned} \int_0^\infty [F_\delta(t, x)]^p dm_\lambda(x) & \leq \int_0^\infty P_t^{[\lambda]}([F_\delta(0, \cdot)]^p)(x) dm_\lambda(x) \\ & \lesssim \int_0^\infty |F_\delta(0, x)|^p dm_\lambda(x) \\ & \sim \int_0^\infty [|f \#_\lambda \phi_\delta(x)| + |R_{\Delta_\lambda}(f \#_\lambda \phi_\delta)(x)|]^p dm_\lambda(x) \lesssim 1. \end{aligned} \quad (3.21)$$

We claim that for each $t, x \in (0, \infty)$, $F_\delta(t, x) \rightarrow F(t, x)$ as $\delta \rightarrow 0$, where

$$F(t, x) \equiv \|\varphi\|_{L^1((0, \infty), dm_\lambda)} \{[P_t^{[\lambda]}(f)(x)]^2 + [Q_t^{[\lambda]}(f)(x)]^2\}^{1/2}.$$

Indeed, observe that for any fixed $x \in (0, \infty)$, $P_t^{[\lambda]}(x, \cdot), Q_t^{[\lambda]}(x, \cdot) \in \mathcal{G}(1, 1)$. Thus we only need to show that for all $\varphi \in Z^{[\lambda]}$, $f \#_\lambda \varphi_\delta \rightarrow f \|\varphi\|_{L^1((0, \infty), dm_\lambda)}$ in $(\mathcal{G}(1, 1))'$ as $\delta \rightarrow 0$. To this end, let $\psi \in \mathcal{G}(1, 1)$. Then by (2.1), we have that when $\delta \rightarrow 0$,

$$\begin{aligned} & \int_0^\infty f \#_\lambda \varphi_\delta(x) \psi(x) dm_\lambda(x) \\ & = \int_0^\infty \int_0^\infty \int_0^\infty D(x, y, z) \varphi_\delta(z) dm_\lambda(z) f(y) dm_\lambda(y) \psi(x) dm_\lambda(x) \\ & = \int_0^\infty \int_0^\infty \int_0^\infty D(x, y, z) \psi(x) dm_\lambda(x) \varphi_\delta(z) dm_\lambda(z) f(y) dm_\lambda(y) \\ & = C_\lambda \int_0^\infty \int_0^\infty \int_0^\pi \delta^{-2\lambda-1} \psi(\sqrt{y^2 + z^2 - 2yz \cos \theta}) \\ & \quad \times \varphi\left(\frac{z}{\delta}\right) (\sin \theta)^{2\lambda-1} d\theta dm_\lambda(z) f(y) dm_\lambda(y) \\ & = C_\lambda \int_0^\infty \int_0^\infty \int_0^\pi \psi(\sqrt{y^2 + \delta^2 z^2 - 2\delta y z \cos \theta}) \varphi(z) (\sin \theta)^{2\lambda-1} d\theta dm_\lambda(z) f(y) dm_\lambda(y) \\ & \rightarrow C_\lambda \int_0^\infty \int_0^\infty \int_0^\pi \psi(y) \varphi(z) (\sin \theta)^{2\lambda-1} d\theta dm_\lambda(z) f(y) dm_\lambda(y) \\ & = \|\varphi\|_{L^1((0, \infty), dm_\lambda)} \int_0^\infty \psi(y) f(y) dm_\lambda(y), \end{aligned}$$

where $C_\lambda \equiv \frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda)\sqrt{\pi}}$. This implies the claim.

By this claim together with (3.21) and the Fatou lemma, we further have that

$$\sup_{0 < t < \infty} \int_0^\infty |F(t, x)|^p dm_\lambda(x) \lesssim 1,$$

from which together with Lemma 3.4, we deduce that

$$\int_0^\infty \sup_{0 < t < \infty} \left| P_t^{[\lambda]}(f)(x) \right|^p dm_\lambda(x) \lesssim 1.$$

Therefore, by Theorem 1.1, we have $f \in H^p((0, \infty), dm_\lambda)$ and hence complete the proof of Theorem 1.2. \square

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